A Formulation of Second-Order Matching

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$$u_1 = (5 + 4) \times (7 + 9) \quad u_2 = 5 \times 7 + 4 \times 7 + 5 \times 9 + 4 \times 9$$

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let rec dist e = match e with
  | Mul(x, Add(y, z)) | Mul(Add(y, z), x) ->
      let x' = dist x in
      Add(Mul(x', dist y),Mul(dx, dist z))
  | Mul(x, y) -> Mul(dist x, dist y)
  | Div(x, y) -> Div(dist x, dist y)
  | Add(x, y) -> Add(dist x, dist y)
  | Sub(x, y) -> Sub(dist x, dist y)
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Theoretical foundation of pattern matching

Given two first-order terms t and u:

- **Unification problem:** find σ such that $t[\sigma] = u[\sigma]$.
- **Matching problem:** find σ such that $t[\sigma] = u$ (u closed).

Example

$t = \texttt{Mul}(\texttt{x}, \texttt{Add}(\texttt{y}, \texttt{z}))$

$\sigma = \{x \mapsto \texttt{Add}(5, 4), y \mapsto 7, z \mapsto 9\}$

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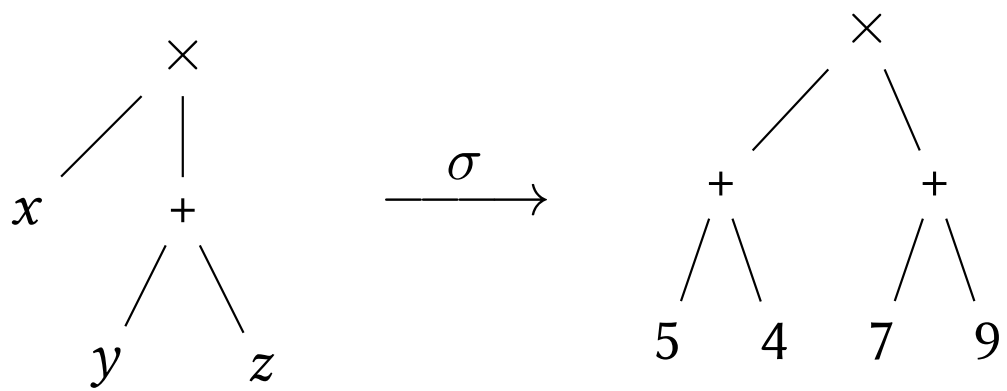

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Theorem (Robinson, 1965)

First-order unification is decidable, and when a solution exists, there exists a unique most general solution that is computable.

Binding

What if we want to add a sum operator to our **expr** language?

$$u_1 = \sum_{i=1}^{5} \sum_{j=1}^{16} i \times i \qquad u_2 = (1 + (16 - 1)) \times \sum_{i=1}^{16} i \times i$$

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Idea 1: use first-order terms

```
type expr = Int of int | Add of expr * expr | ...
          | Var of string
          | Sum of expr * expr * string * expr
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E.g. u_1 is represented by

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Sum(1, 5, "i", Sum(1, 16, "j", Mul(Var "i", Var "i"))).
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- Pattern matching cannot express symbolic binding.
- Similar, repetitive algorithms to handle symbolic binding, such as substitution, equality modulo renaming bound variables, etc.

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type expr = Int of int | Add of expr*expr | ...
          | Var of string (* implicitly defined *)
          | Sum of expr * expr * (expr => expr)
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match u1 with (* try to optimize nested sums *)
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- u_1 matches the pattern, becomes u_2

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Theorem (Huet, 1976)

Second-order matching is decidable.

Goals in this TRE

- Reformulate Huet and Lang (1978)'s second-order matching algorithm in a declarative way.
- Give concise and complete proofs of correctness and termination.
- Implement the algorithm in OCaml.

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Non-goals:

- Design a programming language implementing second-order matching.

Second-order abstract syntax (1/3)

Types: $\tau, \upsilon ::= \mathbb{T} \mid \tau \to \upsilon$.

First-order types: $\mathbb{T}, \mathbb{T} \to \mathbb{T}, \mathbb{T} \to \mathbb{T} \to \mathbb{T}, \ldots$

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Contexts: A context Γ is a set of first-order typed *variables*

Example

$\Gamma = \{x : \mathbb{T},\ f : \mathbb{T} \to \mathbb{T}\}$

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Second-order abstract syntax (3/3)

Substitution: a set $\{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$, denoted σ or ρ.
Applying a substitution to a term hereditarily: replacing each each free occurrence of an x_i in t with t_i, and apply the arguments it had.

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Definition

A closed substitution σ_1 is *more general* than σ_2 if $\sigma_1 \subseteq \sigma_2$.

Matching problem

Definition (Matching problem)

A *matching problem* is a triple (t, u, τ), denoted $t \leq^{\tau}_{?} u$, such that $\Gamma \vdash t : \tau$ and $\emptyset \vdash u : \tau$.

Definition (Solution)

We call *solution* to the problem $t \leq^{\tau}_{?} u$ any closed substitution σ such that $t[\sigma] = u$. We denote the set of solutions for by $\mathcal{F}_{t \leq^{\tau}_{?} u}$.

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Example

Given the problem $f(A, g(B)) \leq^{\mathbb{T}}_{?} A$, possible solutions are

$$\sigma_1 = \{f \mapsto \lambda x\, y.A\}$$

$$\sigma_2 = \{f \mapsto \lambda x\, y.A, g \mapsto \lambda x.x\}$$

$$\sigma_3 = \{f \mapsto \lambda x\, y.x\}$$

σ_1 is more general than σ_2, while σ_1 and σ_3 cannot be compared.

Solutions to a matching problem

Remark

Given the problem $f(A, g(B)) \leq^{\mathbb{T}}_{?} A$, for any closed term t, $\sigma_t = \{f \mapsto \lambda x\, y.x, g \mapsto t\}$ is a solution. But this is artificial: these solutions are induced by the general $\sigma = \{f \mapsto \lambda x\, y.x\}$.

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Lemma (Canonical solution)

Let $\sigma \in \mathcal{F}_{t \leq^{\tau}_{?} u}$. There exists a unique $\mathrm{c}(\sigma) \in \mathcal{F}_{t \leq^{\tau}_{?} u}$ such that $\mathrm{c}(\sigma) \subseteq \sigma$ and $\mathrm{c}(\sigma)$ is minimal for $\subseteq$ in $\mathcal{F}_{t \leq^{\tau}_{?} u}$. If $\mathrm{c}(\sigma) = \sigma$ we say that σ is canonical.

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Definition (Canonical solutions set)

We define $\mathcal{S}_{t \leq^{\tau}_{?} u} ::= \{\mathrm{c}(\sigma) \mid \sigma \in \mathcal{F}_{t \leq^{\tau}_{?} u}\}$

Matching inference system (1/2)

$$\boxed{\Gamma \mid \Delta \vdash t \leq^{\tau}_{?} u \dashv \sigma}$$

1) $t = \lambda x.t'$: add x to the context Δ of bound variables.

$$\textsc{Introduce}\ \frac{\Gamma \mid \Delta, x : \mathbb{T} \vdash t \leq^{\tau}_{?} u \dashv \sigma}{\Gamma \mid \Delta \vdash \lambda x.t \leq^{\mathbb{T}\to\tau}_{?} \lambda x.u \dashv \sigma}$$

Matching inference system (1/2)

$$\boxed{\Gamma \mid \Delta \vdash t \leq^{\tau}_{?} u \dashv \sigma}$$

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2) $t = a(t_1, \ldots, t_n)$ where a is an operator or a bound variable:

- Check that u is of the form $a(u_1, \ldots, u_n)$.
- Find solution σ_1 to $t_1 \leq^{\tau_1}_{?} u_1$.
- Find solution σ_2 to $t_2[\sigma_1] \leq^{\tau_2}_{?} u_2$.
- ...

$$\textsc{Simplify}\ \frac{\begin{array}{c}\Sigma \cup \Delta \ni a : \tau_1 \times \ldots \tau_n \to \mathbb{T} \\ (\Gamma \mid \Delta \vdash t_i[\sigma_{i-1} \circ \ldots \circ \sigma_1]_{\mathsf{h}} \leq^{\tau_i}_{?} u_i \dashv \sigma_i)_{1 \leq i \leq n}\end{array}}{\Gamma \mid \Delta \vdash a(t_1, \ldots, t_n) \leq^{\mathbb{T}}_{?} a(u_1, \ldots, u_n) \dashv \sigma_n \circ \ldots \circ \sigma_1}$$

Matching inference system (2/2)

$$\boxed{\Gamma \mid \Delta \vdash t \leq^{\tau}_{?} u \dashv \sigma}$$

3) $t = f(t_1, \ldots, t_n)$:

- Choose a value for the variable f.
- Substitute it, then solve the remaining problem.

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Intuition

The choice for f must be closed and of the form $\lambda x_1 \ldots x_n.b(v_1, \ldots, v_m)$.

Matching inference system (2/2)

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3) $t = f(t_1, \ldots, t_n)$:

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$$\textsc{Project}\ \frac{\theta := \{f \mapsto \lambda x_1 \ldots x_n.x_i\} \qquad 1 \leq i \leq n \qquad \Gamma \mid \Delta \vdash f(t_1, \ldots, t_n)[\theta]_{\mathsf{h}} \leq^{\tau}_{?} u \dashv \sigma}{\Gamma, f : \_ \mid \Delta \vdash f(t_1, \ldots, t_n) \leq^{\mathbb{T}}_{?} u \dashv \sigma \circ \theta}$$

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- Either b is one of $x_1, \ldots, x_n$: we Project

Matching inference system (2/2)

$$\boxed{\Gamma \mid \Delta \vdash t \leq^{\tau}_{?} u \dashv \sigma}$$

3) $t = f(t_1, \ldots, t_n)$:

- Choose a value for the variable f.
- Substitute it, then solve the remaining problem.

$$\textsc{Imitate}\ \frac{\theta := \{f \mapsto \lambda x_1 \ldots x_n.F(\overrightarrow{\lambda y_1, \ldots, y_{k_i}.f_i(x_1, \ldots, x_n, y_1, \ldots, y_{k_i})})\} \quad \Gamma, f_1 : \_, \ldots, f_m : \_ \mid \Delta \vdash f(t_1, \ldots, t_n)[\theta]_{\mathsf{h}} \leq^{A}_{?} F(u_1, \ldots, u_m) \dashv \sigma}{\Gamma, f : \_ \mid \Delta \vdash f(t_1, \ldots, t_n) \leq^{A}_{?} F(u_1, \ldots, u_m) \dashv (\sigma \circ \theta) - f_1, \ldots, f_m}$$

Intuition

The choice for f must be closed and of the form $\lambda x_1 \ldots x_n.b(v_1, \ldots, v_m)$.

- Either b is one of $x_1, \ldots, x_n$: we Project
- Or b is an operator $F \in \Sigma$: we Imitate, and introduce m variables for the v_i's

Soundness and completeness of the system

Theorem (Soundness)

If we have $\Gamma \mid \emptyset \vdash t \leq^{\tau}_{?} u \dashv \sigma$, *then* σ *is a canonical solution to the problem* $t \leq^{\tau}_{?} u$.

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For all $\sigma \in \mathcal{S}_{t \leq^{\tau}_{?} u}$, *we have* $\Gamma \mid \emptyset \vdash t \leq^{\tau}_{?} u \dashv \sigma$.

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Intuition for the proof of completeness

- We reason by well-founded induction on the pair (u, t) equipped with the lexicographic order $\prec \times_{\mathcal{L}} \prec$ where $\prec$ is the structural order on terms; and we proceed by cases on t.
- For each case and each σ, we choose a rule (e.g. SIMPLIFY) and express σ as a some $e(\sigma_1, \ldots, \sigma_n)$ where the σ_i are solutions to a smaller problem appearing as a premise in the rule.

The inference system is an algorithm

Judgement input and judgement output

$$\underbrace{\Gamma \mid \Delta \vdash t \leq^{\tau}_{?} u}_{input} \dashv \underbrace{\sigma}_{output}$$

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For all the rules (Introduce, Simplify, Project, Imitate):

- Input of the premises $\Leftarrow^{\text{lin}}$ input of the result.
- Output of the result $\Leftarrow^{\text{lin}}$ output of the premises.

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$\Longrightarrow$ The system specifies a recursive nondeterministic algorithm.

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Theorem (Termination)

The algorithm specified by the system terminates.

The inference system is an algorithm

Judgement input and judgement output

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For all the rules (Introduce, Simplify, Project, Imitate):

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Corollary

The set $\mathcal{S}_{t \leq^{\tau}_{?} u}$ is computable.

Conclusion

- We have exposed a simple structure of the set of solutions.
- We presented a second-order matching algorithm as an inference system made of mostly atomic rules.

Future work

- Improve the inference system: reformulate the IMITATE rule using smaller, atomic rules.
- Study a functional language implementing the construct.

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Differences with report

- filter for a problem $\mapsto$ solution to a problem
- solution to a problem $\mapsto$ complete+minimal set of solutions
- Monosorted : only one primitive type $\mathbb{T}$

Misc

Complexity: NP-complete in general. (Lewis D. Baxter, The Complexity of Unification, Ph.D. Thesis, University of Waterloo, 1976)